IMPLICATIONS OF THE HASSE PRINCIPLE FOR ZERO CYCLES OF DEGREE ONE ON PRINCIPAL HOMOGENEOUS SPACES

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Abstract. Let k be a perfect field of virtual cohomological dimension ≤ 2. Let G be a connected linear algebraic group over k such that G^{sc} satisfies a Hasse principle over k. Let X be a principal homogeneous space under G over k. We show that if X admits a zero cycle of degree one, then X has a k-rational point.

Introduction

The following question of Serre [11, pg 192] is open in general.

> **Q:** Let k be a field and G a connected linear algebraic group defined over k. Let X be a principal homogeneous space under G over k. Suppose X admits a zero cycle of degree one, does X have a k-rational point?

Let k be a number field, let V be the set of places of k and let k_v denote the completion of k at a place v. We say that a connected linear algebraic group G defined over k satisfies a *Hasse principle* over k if the map $H^1(k,G) \to \prod_{v\in V} H^1(k_v,G)$ is injective. Let V_r denote the set of real places of k. If G is simply connected, then by a theorem of Kneser, the Hasse principle reduces to injectivity of the maps $H^1(k,G) \to \prod_{v\in V_r} H^1(k_v,G)$. That this result holds is a theorem due to Kneser, Harder and Chernousov [4] ,[5],[6]. Sansuc used this Hasse principle to show that **Q** has positive answer for number fields.

Let k be any field and Ω the set of orderings of k. For $v \in \Omega$ let k_v denote the real closure of k at v. We say that a connected linear algebraic group G defined over k satisfies a *Hasse principle* over k if the map $H^1(k,G) \to \prod_{v\in\Omega} H^1(k_v,G)$ is injective. It is a conjecture of Colliot-Thélène [2, pg 652] that a simply connected semisimple group satisfies a Hasse principle over a perfect field of virtual cohomological dimension ≤ 2. Bayer and Parimala [2] have given a proof in the case where G is of classical type, type F_4 and type G_2.

Our goal in this paper is to extend Sansuc's result by providing a positive answer to **Q** when k is a perfect field of virtual cohomological dimension ≤ 2 and G^{sc} satisfies a Hasse principle over k. More precisely, we prove the following:

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Theorem 0.1. *Let k be a perfect field of virtual cohomological dimension ≤ 2. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k. If G^{sc} satisfies a Hasse principle over k, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

We obtain the following as a corollary:

Corollary 0.2. *Let k be a perfect field of virtual cohomological dimension ≤ 2. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k. If the simple factors of G^{sc} are of classical type, type F_4 or type G_2 then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

is injective.

Sansuc's proof of a positive answer to **Q** over number fields relies on the surjectivity of the map $H^1(k, \mu) \to \prod_{v\in V_r} H^1(k_v, \mu)$ for μ a finite commutative group scheme. This result is a consequence of the Chebotarev density theorem and does not extend to a general field of virtual cohomological dimension ≤ 2. Even in the case $\mu = \mu_2$, the surjectivity of the map $H^1(k, \mu) \to \prod_{v\in\Omega} H^1(k_v, \mu)$ imposes severe conditions on k like the SAP property. The main content of this paper is to replace the arithmetic in Sansuc's paper with a norm principle over a real closed field.

1. Algebraic Groups

In this section, we review some well-known facts from the theory of algebraic groups and define some notation used in the remainder of the work.

Let k be a field. An *algebraic group* G over k is a smooth group scheme of finite type. A surjective morphism of algebraic groups with finite kernel is called an *isogeny* of algebraic groups. An isogeny $G_1 \to G_2$ is said to be *central* if its kernel is a central subgroup of G_1.

An *algebraic torus* is an algebraic group T such that $T(\bar{k})$ is isomorphic to a product of multiplicative groups $G_{m,\bar{k}}$. A torus T is said to be *quasitrivial* if it is a product of groups of the form $R_{E_i/k}G_m$ where $\{E_i\}_{1\leq i\leq r}$ is a family of finite field extensions of k.

An algebraic group G is called *linear* if it is isomorphic to a closed subgroup of GL_n form some n, or equivalently, if its underlying algebraic variety is affine. Of particular interest among connected linear algebraic groups are semisimple groups and reductive groups.

A connected linear algebraic group is called *semisimple* if it has no nontrivial, connected, solvable, normal subgroups. A semisimple group G is said to be *simply connected* if every central isogeny $G' \to G$ is an isomorphism. We can associate to any semisimple group a simply connected group $\tilde{G}$ (unique up to isomorphism)

such that there is a central isogeny $\tilde{G} \to G$. We refer to $\tilde{G}$ as the *simply connected cover* of G.

Any simply connected semisimple group is a product of simply connected simple algebraic groups [7, Theorem 26.8]. Any simple algebraic group belongs to one of four infinite families A_n, B_n, C_n, D_n or is of type E_6, E_7, E_8, F_4 or G_2 (see for example [7, §26]). A simple group which is of type A_n, B_n, C_n or D_n but not of type trialitarian D_4 is said to be a *classical group*. All other simple groups are called *exceptional groups*.

A connected linear algebraic group is called *reductive* if it has no nontrivial, connected, unipotent, normal subgroups. Given a connected linear algebraic group G, the *unipotent radical* of G denoted G^u is the maximal connected unipotent normal subgroup of G. It is clear that G/G^u is always a reductive group. We denote G/G^u by G^{red}. The commutator subgroup of G^{red} is a semisimple group which we denote G^{ss}. We denote the simply connected cover of G^{ss} by G^{sc}.

A *special covering* of a reductive group G is an isogeny

$$1 \to \mu \to G_0 \times S \to G \to 1$$

where G_0 is a simply connected semisimple algebraic k-group and S is a quasitrival k-torus. Given a reductive group G there exists an integer n and a quasitrival torus T such that $G^n \times T$ admits a special covering [9, Lemme 1.10].

2. Galois Cohomology and Zero Cycles

For our convenience, we will discuss **Q** in the context of Galois Cohomology. We briefly review some of the notions from Galois Cohomology we will use and then restate **Q** in this setting.

Let k be a field and $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of k. For an algebraic k-group G, let $H^i(k, G) = H^i(\Gamma_k, G(\bar{k}))$ denote the Galois Cohomology of G with the assumption $i \leq 1$ if G is not abelian. For any k-group G, $H^0(k, G) = G(k)$ and $H^1(k, G)$ is a pointed set which classifies the isomorphism classes of principal homogeneous spaces under G over k. The point in $H^1(k, G)$ corresponds to the principal homogeneous space with rational point. We will interchangeably denote the point in $H^1(k, G)$ by *point* or 1.

Each Γ_k-homomorphism $f : G \to G'$ induces a functorial map $H^i(k, G) \to H^i(k, G')$ which we shall also denote by f. Given an exact sequence of k-groups,

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 1$$

there exists a connecting map $\delta_0 : G_3(k) \to H^1(k, G_1)$ such that the following is an exact sequence of pointed sets.

$$G_1(k) \xrightarrow{f_1} G_2(k) \xrightarrow{f_2} G_3(k) \xrightarrow{\delta_0} H^1(k, G_1) \xrightarrow{f_1} H^1(k, G_2) \xrightarrow{f_2} H^1(k, G_3)$$

If G_1 is central in G_2, there is in addition a connecting map $\delta_1 : H^1(k, G_3) \to H^2(k, G_1)$ such that the following is an exact sequence of pointed sets.

$$G_3(k) \xrightarrow{\delta_0} H^1(k, G_1) \xrightarrow{f_1} H^1(k, G_2) \xrightarrow{f_2} H^1(k, G_3) \xrightarrow{\delta_1} H^2(k, G_1)$$

Given a field extension L of k, $\mathrm{Gal}(\bar{k}/L) \subset \mathrm{Gal}(\bar{k}/k)$ and there is a restriction homomorphism $\mathrm{res} : H^1(k, G) \to H^1(L, G)$. If G is a commutative group, and if the degree of L over k is finite, there is also a corestriction homomorphism

cores : $H^1(L, G) \to H^1(k, G)$. The composition cores $\circ$ res is multiplication by the degree of L over k.

Let p be any prime number. The *p-cohomological dimension* of k is less than or equal to r (written $\mathrm{cd}_p(k) \le r$) if $H^n(k, A) = 0$ for every p-primary torsion Γ_k-module A and $n > r$. The *cohomological dimension* of k is less than or equal to r (written $\mathrm{cd}(k) \le r$), if $\mathrm{cd}_p(k) \le r$ for all primes p. Finally, the *virtual cohomological dimension* of k, written $\mathrm{vcd}(k)$ is precisely the cohomological dimension of $k(\sqrt{-1})$. If k is a field of positive characteristic then $\mathrm{vcd}(k) = \mathrm{cd}(k)$.

Let X be a scheme. For any closed point $x \in X$, let $\mathcal{O}_x$ be the local ring at x and let $\mathfrak{M}_x$ be its maximal ideal. The *residue field* of x written $k(x)$ is $\mathcal{O}_x/\mathfrak{M}_x$. *Zero cycles* of X are elements of the free abelian group on closed points $x \in X$. We may associate to any zero cycle $\sum n_i x_i$ on X its *degree* $\sum n_i[k(x_i) : k]$ where $k(x_i)$ is the residue field of x_i.

A closed point with residue field k is called a *rational point*. It is clear that if x is a closed point of a variety X over k then it is a rational point of $X_{k(x)}$. We have seen that the point in $H^1(*, G)$ is the principal homogeneous space under G over $*$ with a rational point. Therefore, a principal homogeneous space X under G over k, with zero cycle $\sum n_i x_i$ is an element of the kernel of the product of the restriction maps $H^1(k, G) \to \prod H^1(k(x_i), G)$. If the zero cycle is of degree one, then the field extensions $k(x_i)$ are necessarily of coprime degree over k.

Guided by this insight, one may restate **Q** as follows.

Q: Let k be a field and let G be a connected, linear algebraic group defined over k. Let $\{L_i\}_{1 \le i \le m}$ be a collection of finite extensions of k with $\gcd([L_i : k]) = 1$. Does the canonical map

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

have trivial kernel?

3. Orderings of a Field

We recall some basic properties of orderings of a field [10].

An *ordering* ν of a field k is given by a binary relation $\le_\nu$ such that for all $a, b, c \in k$.

- $a \le_v a$
- If $a \le_v b$ and $b \le_\nu c$ then $a \le_v c$
- If $a \le_v b$ and $b \le_v a$ then $a = b$
- Either $a \le_v b$ or $b \le_v a$
- If $a \le_v b$ then $a + c \le_v b + c$
- If $a \le_v b$ and $0 \le_v c$ then $ca \le_v cb$

A field k which admits an ordering is necessarily of characteristic 0. If k is a field with an ordering v, an *algebraic extension* of the ordered field (k, v) is an algebraic field extension L of k together with an ordering v' on L such that v' restricted to k is v. If L is a finite field extension of k of odd degree there is always an algebraic extension (L, v') of (k, v) [10, Chapter 3, Theorem 1.10].

A field k is said to be *formally real* if -1 is not a sum of squares in k. A field k is called a *real closed field* if it is a formally real field and no proper algebraic extension is formally real. There is a unique ordering $\square$ on a real closed field. This

ordering is defined by the relation $a \leq b$ if and only if $b-a$ is a square in k. Further, if k is a real closed field, then $k(\sqrt{-1})$ is algebraically closed [10, Theorem 2.3 (iii)].

If L is a finite field extension of k, then $k_v \otimes L$ is isomorphic to a product of the form $\prod k_v \prod k_v(\sqrt{-1})$. Also, since $k_v(\sqrt{-1})$ is an algebraic closure for k there is a natural inclusion $\mathrm{Gal}(\bar{k}, k_v) \subset \Gamma_k$ and thus a restriction map $H^1(k,G) \to H^1(k_v, G)$.

4. Main Result

In the discussion which follows we will need the following lemmas.

Lemma 4.1. *Let k be a field and let G be a reductive group over k. Fix an integer n and a quasitrivial torus T such that $G^n \times T$ admits a special covering*

$$1 \to \mu \to G_0 \times S \to G^n \times T \to 1$$

Then G^{sc} satisfies a Hasse Principle over k if and only if G_0 satisfies a Hasse principle over k.

Proof. Taking commutator subgroups we have a short exact sequence

$$1 \to \tilde{\mu} \to [G_0 \times S : G_0 \times S] \to [G^n \times T : G^n \times T] \to 1$$

Since S and T are tori, $[G_0 \times S : G_0 \times S] \cong [G_0 : G_0]$ and $[G^n \times T : G^n \times T] = [G^n : G^n]$. That G_0 is semisimple gives $[G_0 : G_0] = G_0$. It is clear that $[G^n : G^n] = [G : G]^n$ which in turn is $(G^{ss})^n$ by definition of G^{ss}. Therefore, we have the following short exact sequence

$$1 \to \tilde{\mu} \to G_0 \to (G^{ss})^n \to 1$$

where $\tilde{\mu}$ is some finite group schme. In particular, G_0 is a simply connected cover of $(G^{ss})^n$. Since $(G^{sc})^n$ is certainly a simply connected cover of $(G^{ss})^n$, uniqueness of the simply connected cover of $(G^{ss})^n$ gives $(G^{sc})^n \cong G_0$. In particular, the simple factors of G^{sc} are the same as the simple factors of G_0 and G^{sc} satisfies the Hasse principle over k if and only if G_0 satisfies the Hasse principle over k. □

Lemma 4.2. *Let k be a real closed field and let G be a reductive group over k which admits a special covering*

$$1 \to \mu \to G_0 \times S \to G \to 1 \tag{4.2.1}$$

Let L be a finite ètale k-algebra. Let δ be the first connecting map in Galois Cohomology and let $N_{L/k}$ denote the corestriction map $H^1(k \otimes L, \mu) \to H^1(k, \mu)$. Then

$$N_{L/k}(\mathrm{im}(G(k \otimes L) \xrightarrow{\delta_L} H^1(k \otimes L, \mu)) \subset \mathrm{im}(G(k) \xrightarrow{\delta} H^1(k, \mu))$$

Proof. Since k is real closed, there exists finite numbers r and s such that $k \otimes L$ is isomorphic to a product of r copies of k and s copies of $k(\sqrt{-1})$. Thus

$$H^1(k \otimes L, \mu) \cong \prod_{r\text{ copies}} H^1(k, \mu) \prod_{s\text{ copies}} H^1(k(\sqrt{-1}), \mu)$$

Since k is real closed, $k(\sqrt{-1})$ is algebraically closed, $H^1(k(\sqrt{-1}), \mu)$ is trivial and $H^1(k \otimes L, \mu)$ is just a product of r copies of $H^1(k, \mu)$. Therefore,

$$N_{L/k} : H^1(k \otimes L, \mu) \to H^1(k, \mu)$$

is just the product map

$$\prod_{r \text{ copies}} H^1(k,\mu) \to H^1(k,\mu)$$

That $k \otimes L$ is a product of r copies of k and s copies of $k(\sqrt{-1})$ also gives that

$$G(k \otimes L) \cong \prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1}))$$

Therefore, the connecting map

$$\prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1})) \xrightarrow{\delta} \prod_{r \text{ copies}} H^1(k,\mu) \prod_{s \text{ copies}} H^1(k(\sqrt{-1}),\mu)$$

is just the product of the connecting maps

$$G(k) \to H^1(k,\mu)$$

and

$$G(k(\sqrt{-1}) \to H^1(k(\sqrt{-1}),\mu)$$

the latter of which is necessarily the trivial map.

So choose

$$(x_1,\dots,x_r,y_1,\dots,y_s) \in G(k \otimes L)$$

Then

$$\begin{aligned} N_{L/k}(\delta(x_1,\dots,x_r,y_1,\dots,y_s)) &= N_{L/k}(\delta(x_1),\dots,\delta(x_r),\delta(y_1),\dots,\delta(y_s)) \\ &= \delta(x_1)\cdots\delta(x_r) \\ &= \delta(x_1\cdots x_r) \end{aligned}$$

Since the x_i were chosen to be in $G(k)$ for all i, then $x_1 \cdots x_r \in G(k)$ and the desired result holds. □

Lemma 4.3. *Let G be a reductive group and L be a finite field extension of k of odd degree. The kernel of the canonical map $H^1(k,G) \to H^1(L,G)$ is contained in the kernel of the canonical map $H^1(k,G) \to \prod_{v\in\Omega} H^1(k_v,G)$.*

Proof. By [10, Chapter 3, Theorem 1.10] each ordering v of k extends to an ordering w of L, in particular each real closure k_v is L_w for some ordering w on L. Since the natural map $H^1(k,G) \to H^1(L_w,G)$ factors through the canonical map $H^1(k,G) \to H^1(L,G)$, the desired result is immediate. □

We now return to the result which is the main goal of this paper.

Theorem 4.4. *Let k be a perfect field of virtual cohomological dimension ≤ 2 and let G be a connected linear algebraic group over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. If G^{sc} satisfies a Hasse principle over k, then the canonical map*

$$H^1(k,G) \to \prod_{i=1}^{m} H^1(L_i,G)$$

has trivial kernel.

Proof. By definition of the groups involved, the following sequence is exact

$$1 \to G^u \to G \to G^{\mathrm{red}} \to 1 \tag{4.4.1}$$

Since G^u is unipotent, $H^i(k, G^u)$ is trivial for $i \geq 1$ and (4.4.1) induces the long exact sequence in Galois Cohomology

$$1 \to H^1(k, G) \to H^1(k, G^{\mathrm{red}}) \to 1$$

which gives that $H^1(k, G) \cong H^1(k, G^{\mathrm{red}})$. Thus to prove 4.4 it is sufficient to consider the case where G is a reductive group. Then fix an integer n and quasitrivial torus T such that $G^n \times T$ admits a special covering

$$1 \to \mu \to G_0 \times S \to G^n \times T \to 1$$

By functoriality, $H^1(k, G^n \times T) \cong H^1(k, G)^n \times H^1(k, T)$ and since T is quasitrivial, $H^1(k, T) = 1$. It follows that our result holds for G if and only if it holds for $G^n \times T$. Replacing G by $G = G^n \times T$ we assume that G admits a special covering

$$1 \to \mu \to G_0 \times S \to G \to 1$$

If k is a field of positive characteristic, $\mathrm{cd}(d) = \mathrm{vcd}(k) = 2$. Since k has no orderings and by hypothesis G^{sc} satisfies a Hasse principle over k then $H^1(k, G^{sc}) = \{1\}$. In particular $H^1(k, G_0) = \{1\}$ and the special covering of G above induces the following commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q"] & H^2(k,\mu) \arrow[d, "r"] \\ 1 \arrow[r] & \prod_i H^1(L_i,G) \arrow[r] & \prod_i H^2(L_i,\mu) \end{tikzcd} \tag{4.4.2}$$

Choose $\lambda \in \ker(q)$. By commutativity of the diagram $h(\lambda) \in \ker(r)$. A restriction-corestriction argument gives r has trivial kernel. Thus $h(\lambda) = \text{point}$. Then by exactness of the top row of the diagram, $\lambda = \text{point}$. (c.f. [3] for the case k a "good" field of cohomological dimension 2.)

Therefore, we may assume that the characteristic of k is zero. Fix an index i. The special covering of G above induces the following commutative diagram with exact rows where the vertical maps are the restriction maps.

$$\begin{tikzcd} H^1(k,\mu) \arrow[r, "f"] \arrow[d] & H^1(k,G_0) \arrow[r, "g"] \arrow[d, "p"] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q"] & H^2(k,\mu) \arrow[d, "r"] \\ \prod H^1(L_i,\mu) \arrow[r, "f"] & \prod H^1(L_i,G_0) \arrow[r, "g"] & \prod H^1(L_i,G) \arrow[r] & \prod H^2(L_i,\mu) \end{tikzcd} \tag{4.4.3}$$

Let λ be in $\ker(q)$. Taking $\mathrm{cores} \circ \mathrm{res}$ we find that r has trivial kernel and thus by commutativity of (4.4.3), λ is in $\ker(h)$. By exactness of the top row, we choose $\lambda' \in H^1(k, G_0)$ such that $g(\lambda') = \lambda$. Write $p(\lambda') = (\lambda'_{L_i})$. Since $g(\lambda'_{L_i}) = \text{point}$, by exactness of the bottom row of (4.4.3) choose $\eta_{L_i} \in H^1(L_i, \mu)$ such that $f(\eta_{L_i}) = \lambda'_{L_i}$.

For each ordering v of k, the special covering of G above also induces the following commutative diagram with exact rows.
(4.4.4)

$$\begin{tikzcd} H^1(k,\mu) \arrow[r, "f"] \arrow[d] & H^1(k,G_0) \arrow[r, "g"] \arrow[d, "p'"] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q'"] & H^2(k,\mu) \arrow[d, "r'"] \\ \prod_{v\in\Omega} H^1(k_v,\mu) \arrow[r, "f"] & \prod_{v\in\Omega} H^1(k_v,G_0) \arrow[r, "g"] & \prod_{v\in\Omega} H^1(k_v,G) \arrow[r] & \prod_{v\in\Omega} H^2(k_v,\mu) \end{tikzcd}$$

By Lemma 4.3, λ is in the kernel of q'. Thus by commutativity of (4.4.4), $(\lambda'_v) = p'(\lambda')$ is in $\ker(g)$. Then by exactness of the bottom row of (4.4.4) choose $\alpha_v \in H^1(k_v,\mu)$ such that $f(\alpha_v) = \lambda'_v$. Let $(\alpha_v)_{L_i}$ denote the image of α_v under the canonical map $H^1(k_v,\mu) \to H^1(k_v \otimes L_i,\mu)$. Let $(\eta_{L_i})_v$ denote the image of η_{L_i} under the canonical map $H^1(L_i,\mu) \to H^1(k_v \otimes L_i,\mu)$.

By choice of α_v and η_{L_i}, $f((\alpha_v)_{L_i}) = (\lambda'_v)_{L_i} = (\lambda'_{L_i})_v = f((\eta_{L_i})_v)$. In particular, $f((\alpha_v)_{L_i}(\eta_{L_i})_v^{-1})$ is the point in $H^1(k_v \otimes L_i, G_0)$. We have a commutative diagram

(4.4.5)
$$\begin{tikzcd} G(k_v) \arrow[r, "\delta"] \arrow[d] & H^1(k_v,\mu) \arrow[r, "f"] \arrow[d] & H^1(k_v,G_0) \arrow[d] \\ \prod_i G(k_v\otimes L_i) \arrow[r, "\delta_{L_i}"] & \prod_i H^1(k_v\otimes L_i,\mu) \arrow[r, "f"] & \prod_i H^1(k_v\otimes L_i,G_0) \end{tikzcd}$$

Exactness of the bottom row of (4.4.5) gives that $(\alpha_v)_{L_i}(\eta_{L_i})_v^{-1}$ is in the image of δ_{L_i}. Choose m_i such that $\sum m_i[L_i:k] = 1$. Since δ_{L_i} is multiplicative, it follows that for each index i, $(\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}$ is in the image of δ_{L_i}

By Lemma 4.2 above, there exists γ_v in $G(k_v)$ such that

$$\delta(\gamma_v) = \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i})$$

Since by restriction-corestriction $N_{L_i/k}((\alpha_v)_{L_i}^{m_i}) = \alpha_v^{m_i[L_i:k]}$. It follows that

$$\begin{aligned} \delta(\gamma_v) &= \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}) \\ &= \alpha_v^{\sum_i m_i[L_i:k]} \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i} \\ &= \alpha_v \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i} \end{aligned}$$

In turn

$$\delta(\gamma_v)\prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i} = \alpha_v$$

Since f is well-defined on the cosets of $G(k_v)$ in $H^1(k_v,\mu)$ [8] and the top row of (4.4.5) is exact, it follows that

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i}\right) = f(\alpha_v)$$

By choice of α_v the latter is λ'_v. Since G^{sc} satisfies a Hasse principle over k, Lemma 4.1 gives that G_0 satisfies a Hasse principle over k. In particular, the map

$H^1(k, G_0) \to \prod_v H^1(k_v, G_0)$ is injective, and since $f(\prod_i (N_{L_i/k}(\eta_{L_i})^{m_i}))_v = \lambda'_v$ for all v, then

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right) = \lambda'$$

Taking g as in (4.4.3) above

$$g\left(f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right)\right) = g(\lambda')$$

Then by exactness of the top row of (4.4.3), $\lambda = g(\lambda')$ =point. □

Applying [2, Theorem 10.1] a Serre twist we obtain the following corollary:

Corollary 4.5. *Let k be a perfect field of virtual cohomological dimension ≤ 2. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k. If the simple factors of G^{sc} are of classical type, type F_4 or type G_2 then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

is injective.

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